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The Translational Hull of Reductive Matrix 0-Bands

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We consider matrix 0-bands $[\gamma]^0$ where γ is a dense relation on a set V. If the semigroup $[\gamma]^0$ is reductive, then every bitranslation of $[\gamma]^0$ is in one-to-one correspondence with a left adjoint map on a so-called weakly ordered set $(W, \lhd)$ with 0 and 1, which is closely related to (V, γ). This interpretation of the translational hull $\Omega([\gamma]^0)$ of a reductive square matrix 0-band is useful for characterizing several semigroups of residuated mappings on bounded posets. In particular, some part of Zareckiĭ's work on binary relations can be seen in this light. This paper thus provides a link between the general results of Petrich on the tranlational hull of Rees matrix semigroups and Zareckiĭ's abstract characterization of certain semigroups of binary relations. © 1986 Academic Press, Inc.

1. INTRODUCTION

For a partially ordered set (poset) W one considers the semigroup $\mathrm{Res}(W)$ of all residuated mappings φ on W, i.e., mappings $\varphi\colon W \to W$ which admit an adjoint $\varphi^+\colon W \to W$ such that $x\varphi \leqq y \Leftrightarrow x \leqq y\varphi^+$ holds throughout. The semigroups $\mathrm{Res}(W)$ play a fundamental rôle in the "semigroup coordinatization of bounded posets," see Blyth and Janowitz [1] and Johnson [2]. It is quite surprising that these semigroups Res (W) have not really been characterized so far, although for bounded posets W this is easy to accomplish: the zero map and all two-valued maps in $\mathrm{Res}(W)$ form a densely embedded 0-minimal ideal of $\mathrm{Res}(W)$. This ideal has a very simple structure: it is a certain matrix 0-band $[\gamma]^0$, where γ is closely related to the partial order of W. Recall that a matrix 0-band (alias rectangular 0-band) is simply a Rees matrix semigroup over the trivial 0-group. The semigroup $\mathrm{Res}(W)$ is then isomorphic to the translational hull of $[\gamma]^0$. This, of course, generalizes the well-known characterization of the semigroup $\mathscr{B}(X)$ of all binary relations on a set X, see Zareckiĭ [13, 16]. Indeed, $\mathscr{B}(X)$ coincides with $\mathrm{Res}(\underline{2}^X)$, where $\underline{2}^X$ denotes the lattice of all subsets of X.

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More generally, the characterization of the semigroup $\mathrm{Res}(W)$ can be performed in the same way for "weakly ordered sets W with 0 and 1." Roughly speaking, a weak order $\leqq$ is a binary relation which guarantees that for every adjoint pair (φ, φ^+) defined as above each map determines the other uniquely (which is, of course, true with partial orders). Recall that a semigroup S is called *reductive* if for any $a, b \in S$, $sa = sb$ for all $s \in S$ implies $a = b$, and $as = bs$ for all $s \in S$ implies $a = b$. We then have the following result: the semigroups $\mathrm{Res}(W)$ of weakly ordered sets W with 0 and 1 are up to isomorphism the translational hulls of reductive square matrix 0-bands, and moreover, the translational hull of any reductive matrix 0-band can be represented as a maximal submonoid of some $\mathrm{Res}(W)$.

The translational hull of an arbitrary Rees matrix semigroup with 0 was constructed by M. Petrich [4]. In [5, 8] he pointed out that some results of Zareckiĭ [13, 16] can be understood in this context. Our description of the translational hull of a reductive Rees matrix semigroup over the trivial 0-group represents an intermediate case, which is more appropriate to the characterization of $\mathscr{B}(X)$ and its maximal submonoids.

The material is organized as follows. Sections 2 and 3 provide some basic facts concerning weakly ordered sets $(W, \lhd)$ and their semigroups $\mathrm{Res}(W, \lhd)$ of left adjoint maps. In Section 4 we briefly discuss the (obvious) relationship of reductive square matrix 0-bands and weakly ordered sets with distinguished elements 0 and 1. Sections 5, 6 contain the main results: the characterizations of the semigroups $\mathrm{Res}(W, \lhd)$ and their maximal submonoids. In the final section we apply the preceding results to obtain abstract characterizations of (i) the semigroups $\mathscr{B}(X)$ [13] and (ii) their maximal submonoids $\mathscr{B}_\alpha(X)$ [16], (iii) the semigroups $\mathscr{D}(X)$ of all dense relations (cf. [15]) and (iv) their maximal submonoids.

Most (standard) semigroup concepts (such as "translational hull," etc.) are not explained in this paper; the reader is referred to the survey article [6] and the book [7] of Petrich.

2. Weakly Ordered Sets

A binary relation $\lhd$ on a (nonempty) set W is called a *weak order* if it satisfies the following two conditions:

(1) if $a \lhd x \Leftrightarrow b \lhd x$ for all x, then $a = b$;

(2) if $x \lhd a \Leftrightarrow x \lhd b$ for all x, then $a = b$.

Then $(W, \lhd)$ is called a weakly ordered set (*woset*). If $(W, \lhd)$ is a woset, then $(W, \not\lhd)$ and $(W, \rhd)$ are wosets, too, where $\not\lhd$ and $\rhd$ denote the complement and the converse of the relation $\lhd$, respectively. The notion of a weak order is fairly general; it covers, for instance, all reflexive antisymmetric relations. In particular, every poset $(W, \leqq)$ is a woset. There is a simple way to derive new weak orders from a given one: let $(W, \lhd)$ be a woset and let θ_1, θ_2 be any bijections of a set V onto W; then define a binary relation $\lhd'$ on V by the rule

(3) $x \lhd' y \Leftrightarrow x\theta_1 \lhd y\theta_2$ $(x, y \in V)$.

It is easy to see that $\lhd'$ is a weak order on V. The wosets $(W, \lhd)$ and $(V, \lhd')$ are then called *matrix isomorphic* (i.e., $\lhd$ and $\lhd'$ are isomorphic relations in the sense of [10]). If one can choose a bijection $\theta_1 = \theta_2$ such that (3) holds, then $(W, \lhd)$ and $(V, \lhd')$ are *isomorphic* wosets. Even with reflexive antisymmetric wosets, "matrix isomorphic" is in general strictly weaker than "isomorphic." However, we do have that *a poset* $(W, \lhd)$ *and a reflexive antisymmetric woset* $(V, \lhd')$ *are matrix isomorphic if and only if they are isomorphic.* To see that this is so, let $\lhd'$ and $\lhd$ be related via (3); then for $x \in V$, $x \lhd' x$ and $x\theta_2\theta_1^{-1} \lhd' x\theta_2\theta_1^{-1}$, whence $x\theta_1 \lhd x\theta_2 \lhd x\theta_2\theta_1^{-1}\theta_2$ and thus by transitivity, $x \lhd' x\theta_2\theta_1^{-1}$. Since also $x\theta_2\theta_1^{-1} \lhd' x$ (because $x\theta_2 \lhd x\theta_2$), we must have $x = x\theta_2\theta_1^{-1}$ for all x by antisymmetry. Hence $\theta_1 = \theta_2$ is an isomorphism of V onto W, completing the proof.

Natural examples of symmetric weak orders are provided by the orthogonality relations $\perp$ of orthomodular lattices or, more generally, of involution posets (in the sense of [1, Sect. 18]). In fact, this is a special instance of (3): let $\theta_1: x \mapsto x$ be the identity map on $V = W$ and let $\theta_2: x \mapsto x'$ be an involution (i.e., $x = x''$ for all $x \in W$) which is antitone with respect to $\lhd$, that is, $x \lhd y \Rightarrow y' \lhd x'$ for all x, y. In this case we say that $(W, \lhd, ')$ is an *involution woset.* Then the relation defined by (3) is symmetric, and we therefore use the symbol $\perp: x \perp y \Leftrightarrow x \lhd y'$ $(x, y \in W)$. It is obvious that *this establishes a one-to-one correspondence between involution wosets* $(W, \lhd, ')$ *and symmetric wosets* $(W, \perp)$ *with an (arbitrary) involution* $x \mapsto x'$.

The wosets under consideration will usually have distinguished elements 0 and 1, subject to the condition

(4) $0 \lhd x$ and $x \lhd 1$ for all x.

If 0 and 1 exist, then they are unique by virtue of (1) and (2). If 0 and 1 are equal (e.g., if $\lhd$ is symmetric) then we prefer to write ∞ instead of $0 = 1$ and call $(W, \lhd)$ a woset with ∞. Note that *every woset* $(W, \lhd)$ *with* 0 *and* 1 *is matrix isomorphic to a woset* $(W, \lhd')$ *with* ∞. Indeed, in (3) let θ_1 be the identity map, and choose any bijection θ_2 which transforms 0 into 1. By a *bounded* woset we mean a woset with 0 and 1 such that $0 \neq 1$ and, in addition,

(5) $x \not\lhd 0$ and $1 \not\lhd y$ for all $x \neq 0$, $y \neq 1$.

We will next show that every woset with ∞ can be transformed into a bounded woset, and vice versa. First, note that to any woset $(V, \lhd)$ one can adjoin two new elements 0, 1 and extend $\lhd$ to $V \cup \{0, 1\}$ via (4) and (5) such that $V^{01} = (V \cup \{0, 1\}, \lhd)$ becomes a bounded woset. If a woset $(V, \lhd)$ has neither 0 nor 1, then adjunction of ∞ yields a woset $V^{\infty} = (V \cup \{\infty\}, \lhd)$ with ∞. Now, given a bounded woset $(W, \lhd)$, first remove 0 and 1 to obtain a woset $(W - \{0, 1\}, \lhd)$. Then for the complement $\not\!\lhd$ of $\lhd$ on $W - \{0, 1\}$ there exist no zero and no one element by virtue of (1), (2), (5). Hence $(W - \{0, 1\}, \not\!\lhd)^{\infty}$ is a woset with ∞. On the other hand, given a woset $(V, \lhd')$ with ∞, drop the element ∞, take the complement of $\lhd'$, and then add 0 and 1, so that one obtains a bounded woset $(V - \{\infty\}, \not\!\lhd')^{01}$. Trivially, these two constructions are mutually inverse.

3. Adjoint Maps of Wosets

Let $(W, \lhd)$, $(V, \lhd')$ be wosets, and let $\varphi\colon V \to W$ and $\varphi^+\colon W \to V$ be mappings such that for any $x \in V$ and $y \in W$, $x\varphi \lhd y$ if and only if $x \lhd' y\varphi^+$. Then (φ, φ^+) is called an *adjoint pair.* Conditions (1), (2) ensure that φ and φ^+ uniquely determine each other. Notice that (via (3)) any matrix isomorphic wosets W and V give rise to an adjoint pair $(\theta_1, \theta_2^{-1})$; thus, the *matrix isomorphism* $\theta_1\colon (V, \lhd') \to (W, \lhd)$ uniquely determines its companion θ_2. Henceforth we will only be interested in adjoint pairs (φ, φ^+) defined on one and the same woset $(W, \lhd)$, i.e.,

(6) $x\varphi \lhd y \Leftrightarrow x \lhd y\varphi^+$ for all x, y.

For an adjoint pair (φ, φ^+), φ is called a *left adjoint map* with φ^+ as its *right adjoint.* In the poset case, φ and φ^+ are also said to be residuated and residual, respectively. If φ and ψ are left adjoint maps, so is their composition $\varphi\psi$, and we have $(\varphi\psi)^+ = \psi^+\varphi^+$. Consequently, the left adjoint maps on $(W, \lhd)$ form a semigroup $\mathrm{Res}(W, \lhd)$ (for posets briefly written as $\mathrm{Res}(W)$). The correspondence $\varphi \mapsto \varphi^+$ sets up an anti-isomorphism between $\mathrm{Res}(W, \lhd)$ and the semigroup $\mathrm{Res}^+(W, \lhd)$ of right adjoint maps. $\mathrm{Res}(W, \lhd)$ contains, for instance, all matrix automorphisms (i.e., bijective left adjoint maps) and hence all automorphisms. If φ and φ^+ are any mappings satisfying

(7) $x \lhd x\varphi\varphi^+$ and $x\varphi^+\varphi \lhd x$ for all x,

(8) if $x \lhd y$, then $x\varphi \lhd y\varphi$ and $x\varphi^+ \lhd y\varphi^+$,

then (φ, φ^+) is an adjoint pair provided that $\lhd$ is transitive. Indeed, if $x\varphi \lhd y$, then $x \lhd x\varphi\varphi^+$ and $x\varphi\varphi^+ \lhd y\varphi^+$ by (7) and (8), whence $x \lhd y\varphi^+$; similarly, $x \lhd y\varphi^+$ implies $x\varphi \lhd y$. On the other hand, if $\lhd$ is a partial order, then every adjoint pair (φ, φ^+) satisfies (7) and (8). To see this, observe that (7) is immediate by reflexivity, and then (8) follows from $x\varphi^+\varphi \lhd x \lhd y \lhd y\varphi\varphi^+$ by transitivity. Hence we arrive at the following fact (see J. Schmidt [12]): *a pair* (φ, φ^+) *of mappings on a poset is adjoint if and only if it satisfies* (7) *and* (8). Therefore left adjoint maps on a poset W coincide with residuated maps and their right adjoints coincide with residual maps in the sense of Blyth and Janowitz [1].

If $(W, \lhd)$ *and* $(V, \lhd')$ *are matrix isomorphic wosets, then the semigroups* $\operatorname{Res}(W, \lhd)$ *and* $\operatorname{Res}(V, \lhd')$ *are isomorphic.* For, suppose that (3) holds. Then any adjoint pair (φ, φ^+) in $(W, \lhd)$ gives an adjoint pair $(\theta_1\varphi\theta_1^{-1}, \theta_2\varphi^+\theta_2^{-1})$ in $(V, \lhd')$, and vice versa. In particular, if $(W, \lhd, ')$ is an involution woset and $(W, \perp)$ is the associated symmetric woset, then $\operatorname{Res}(W, \lhd) = \operatorname{Res}(W, \perp)$ and the right adjoint φ^* of φ in $(W, \perp)$ is given by $x\varphi^* = x'(\varphi^+)'$ for all x (for involution posets, cf. [1, Sect. 17]).

From now on we assume that the wosets W under consideration always have 0 and 1. If (φ, φ^+) is an adjoint pair, then, necessarily, $0\varphi = 0$ and $1\varphi^+ = 1$. In general, 1φ and $0\varphi^+$ are different from 1 and 0, respectively. If, however, $1\varphi = 1$ and $0\varphi^+ = 0$ is true, then φ is said to be *bounded*; the semigroup of all bounded left adjoint maps on a bounded woset $(W, \lhd)$ is denoted by $\operatorname{Res}_{01}(W, \lhd)$. It is easy to see that a left adjoint map φ on a bounded woset is bounded if and only if $1\varphi = 1$ and $x\varphi \neq 0$ for all $x \neq 0$. For future purposes we record here:

LEMMA 1. *The semigroup* $\operatorname{Res}_{01}(W, \lhd)$ *of a bounded woset* $(W, \lhd)$ *is isomorphic to the semigroup* $\operatorname{Res}(V, \lhd')$ *of the associated woset* $(V, \lhd')$ *with* ∞.

Proof. Recall that a bounded woset $(W, \lhd)$ and a woset $(V, \lhd')$ with ∞ are associated if and only if $W - \{0, 1\} = V - \{\infty\}$ and the restrictions of $\lhd$ and $\lhd'$ to this set are complementary to each other. Now, for associated objects W and V the maps $\varphi \in \operatorname{Res}_{01}(W, \lhd)$ and $\psi \in \operatorname{Res}(V, \lhd')$ correspond to each other as one expects: on $W - \{0, 1\} = V - \{\infty\}$ we let either $x\varphi = 1$ and $x\psi = \infty$ or $x\varphi = x\psi$ and, similarly, either $y\varphi^+ = 0$ and $y\psi^+ = \infty$ or $y\varphi^+ = y\psi^+$. It is then readily verified that $\varphi \leftrightarrow \psi$ establishes the asserted isomorphism between $\operatorname{Res}_{01}(W, \lhd)$ and $\operatorname{Res}(V, \lhd')$.

The condition that W have 0 and 1 guarantees that there is an ample supply of adjoint maps. In fact, for every pair (a, b) in W with $a \neq 1$ and $b \neq 0$ we get mappings ξ_b^a and $(\xi_b^a)^+$ on W defined by

$$x\xi_b^a = \begin{cases} 0 & \text{if } x \lhd a \\ b & \text{otherwise} \end{cases} \quad \text{and} \quad y(\xi_b^a)^+ = \begin{cases} 1 & \text{if } b \lhd y \\ a & \text{otherwise} \end{cases} \quad (x, y \in W). \tag{9}$$

ξ_b^a and $(\xi_b^a)^+$ form an adjoint pair since for any x, y in W,

$$x\xi_b^a \lhd y \Leftrightarrow \text{either } x \lhd a \text{ or } b \lhd y \Leftrightarrow x \lhd y(\xi_b^a)^+.$$

Conditions (1), (2) ensure that for different pairs (a, b), (c, d) (where $a, c \neq 1$ and $b, d \neq 0$) the corresponding maps ξ_b^a and ξ_d^c are different. Moreover, every ξ_b^a is different form the zero element ξ_0 of $\operatorname{Res}(W, \lhd)$ defined by $x\xi_0 = 0$ and $y\xi_0^+ = 1$ for all x, y.

Let $\varphi \in \operatorname{Res}(W, \lhd)$ be nonzero. Then $\xi_b^a \varphi = \xi_{b\varphi}^a$ and $\varphi \xi_b^a = \xi_b^{a\varphi^+}$. Further, since $\varphi \neq \xi_0$, we can find u, v such that $u\varphi \not\lhd v$, whence $\xi_b^a = \xi_u^a \varphi \xi_b^v$ belongs to the ideal generated by φ. This establishes

LEMMA 2. *For every woset* $(W, \lhd)$ *with* 0 *and* 1, *the set* $M(W, \lhd) = \{\xi_b^a \mid a \neq 1, b \neq 0\} \cup \{\xi_0\}$ *is the* 0*-minimal ideal of* $\operatorname{Res}(W, \lhd)$.

By virtue of Lemma 1 there is an analogous result for the semigroup $\operatorname{Res}_{01}(W)$ of a bounded woset W. The maps ξ_b^a of the associated woset V with ∞ correspond to the maps ζ_b^a on W $(a, b \neq 0, 1)$ where

$$x\zeta_b^a = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } 0 \neq x \lhd a \\ 1 & \text{otherwise} \end{cases} \qquad \text{and}$$

$$y(\zeta_b^a)^+ = \begin{cases} 1 & \text{if } y = 1 \\ a & \text{if } b \lhd y \neq 1 \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in W). \tag{10}$$

The map ζ_1^0 serves as a zero element of the semigroup $\operatorname{Res}_{01}(W, \lhd)$. Hence from Lemma 2 we obtain

LEMMA 3. *For every bounded woset* $(W, \lhd)$, *the set* $M_{01}(W, \lhd) = \{\zeta_b^a \mid a, b \neq 0, 1\} \cup \{\zeta_1^0\}$ *is the* 0*-minimal ideal of* $\operatorname{Res}_{01}(W, \lhd)$.

4. WOSETS VERSUS REDUCTIVE MATRIX 0-BANDS

A relation $\gamma \subseteq A \times B$ between two sets A, B is called *dense* if $A\gamma = B$ and $B\gamma^{-1} = A$, that is, for each $a \in A$ and $b \in B$ there exist $x \in A$ and $y \in B$ such that $a\gamma y$ and $x\gamma b$. A *matrix* 0*-band* $[\gamma]^0$ in the sense of Lallement [3] and Schein [10] is a semigroup with 0 determined by a dense relation $\gamma \subseteq A \times B$ as follows: $[\gamma]^0 = A \times B \cup \{0\}$ and

$$(a, b) \cdot (c, d) = \begin{cases} (a, d) & \text{if } c\gamma b \\ 0 & \text{otherwise} \end{cases} \qquad (a, c \in A, b, d \in B).$$

This semigroup is reductive if and only if $a\gamma = c\gamma \Rightarrow a = c$, and $b\gamma^{-1} = d\gamma^{-1} \Rightarrow b = d$. Recall that a matrix 0-band $[\gamma]^0$ can be regarded as a Rees matrix semigroup $\mathscr{M}^0[A, \{1\}, B; P]$ over the trivial 0-group with a $B \times A$ sandwich matrix P (and vice versa), where P is the Boolean matrix of the relation γ^{-1}. A Boolean matrix P is the sandwich matrix of a reductive Rees matrix semigroup with 0 if and only if (i) no row or column consists entirely of zeros and (ii) no two different rows or columns are equal (cf. [7]).

We say that the matrix 0-band $[\gamma]^0$ is *square* whenever A and B are of equal cardinality. Now, the following fact is evident: *for a square matrix 0-band* $[\gamma]^0$, *where* $A = B$, γ *is a weak order if and only if* $[\gamma]^0$ *is reductive*. In this case, the initial assumption that γ is dense implies that the complement $A^2 - \gamma$ of γ is a weak order without 0 and 1. Thus it is also clear how reductive square matrix 0-bands $[\gamma]^0$ are related to wosets $(W, \lhd)$ with 0 and 1: given $(W, \lhd)$, the semigroup $M(W, \lhd)$ of Lemma 2 is isomorphic to the reductive square matrix 0-bands $[\gamma]^0$ where γ is the restriction of $\not\!\rhd$ to $A \times B = W - \{1\} \times W - \{0\}$; conversely, given $[\gamma]^0$ where $A = B$, $(W, \lhd) = (A, A^2 - \gamma^{-1})^\infty$ is a woset with ∞ such that $M(W, \lhd)$ is isomorphic to $[\gamma]^0$. We summarize the situation in the following

LEMMA 4. *For a semigroup M the following conditions are equivalent*:

(i) *M is isomorphic to a reductive square matrix 0-band* $[\gamma]^0$,

(ii) $M \cong M(W, \lhd)$ *for some woset* $(W, \lhd)$ *with* 0 *and* 1,

(iii) $M \cong M_{01}(W, \lhd)$ *for some bounded woset* $(W, \lhd)$.

5. Abstract Characterization of $\mathrm{Res}(W, \lhd)$

Let $(W, \lhd)$ be a woset with 0 and 1. We wish to show that $M(W, \lhd)$ is a densely embedded ideal of $\mathrm{Res}(W, \lhd)$. Since $M(W, \lhd)$ is reductive, by Gluskin's theorem $M(W, \lhd)$ is densely embedded if and only if $\mathrm{Res}(W, \lhd)$ is isomorphic to the translational hull $\Omega(M(W, \lhd))$ of $M(W, \lhd)$ (cf. [6, 7]). Inasmuch as $M(W, \lhd)$ is a Rees matrix semigroup with 0 there is a description of $\Omega(M(W, \lhd))$ available (see [4, 7]), and it would thus suffice to recognize $\Omega(M(W, \lhd))$ as $\mathrm{Res}(W, \lhd)$. In this special case, however, a direct proof is so simple that we give it right away: let (λ, ρ) be a bitranslation of the matrix 0-band $M = \{(a,b) | a \in W - \{1\}, b \in W - \{0\}\} \cup \{0\} \cong M(W, \lhd)$. For $y \in W - \{1\}$ pick any $z \in W$ such that $z \not\!\rhd y$. Then (y, z) is idempotent, whence $\lambda(y, z)$ is either 0 or of the form (x, z) for a suitable $x \in W - \{1\}$ (it is not difficult to see that the element x is independent of z). We may therefore define a mapping ψ: $W \to W$ by $1\psi = 1$, $y\psi = 1$ if $\lambda(y, z) = 0$, and $(y\psi, z) = \lambda(y, z)$ otherwise.

Similarly, the right translation ρ induces a mapping φ: $W \to W$ defined by $0\varphi = 0$, $x\varphi = 0$ if $(w, x)\,\rho = 0$, and $(w, x\varphi) = (w, x)\,\rho$ otherwise, where $x \neq 0$ and $x \ntriangleleft w$. Since λ and ρ are linked, the condition $(w, x)\,\lambda(y, z) = (w, x)\,\rho(y, z) \neq 0$ is equivalent to either of the conditions $x \ntriangleleft y\psi$ or $x\varphi \ntriangleleft y$, whence for any $x \neq 0$, $y \neq 1$ we get $x \lhd y\psi \Leftrightarrow x\varphi \lhd y$. By definition of φ and ψ this equivalence is also true for $x = 0$ or $y = 1$, and consequently (φ, ψ) is an adjoint pair, that is, $\varphi \in \operatorname{Res}(W, \lhd)$ and $\psi = \varphi^{+}$. It is now easy to see that $(\lambda, \rho) \to \varphi$ establishes the desired isomorphism of $\Omega(M)$ onto $\operatorname{Res}(W, \lhd)$. Hence from Lemma 4 we obtain

THEOREM 1. *The semigroups* $\operatorname{Res}(W, \lhd)$ *of wosets* $(W, \lhd)$ *with* 0 *and* 1 *are up to isomorphism the translational hulls of reductive square matrix* 0-*bands. In particular, given a woset* $(W, \lhd)$ *with* 0 *and* 1 (*or a bounded woset* $(W, \lhd)$, *respectively*), *a semigroup S is isomorphic to* $\operatorname{Res}(W, \lhd)$ (*or* $\operatorname{Res}_{01}(W, \lhd)$, *respectively*) *if and only if S contains a densely embedded ideal isomorphic to the matrix* 0-*band* $[\gamma]^0$, *where* γ *is the restriction of the relation* $\ntriangleright$ *to* $W - \{1\} \times W - \{0\}$ (*or the restriction of* $\rhd$ *to* $(W - \{0, 1\})^2$, *respectively*).

COROLLARY 1. *All semigroups* $\operatorname{Res}(W, \lhd)$ *are subdirectly irreducible.*

Corollary 1 follows from Theorem 1 by means of [8, Lemma 5] (cf. [7, III.5.19]) since $M(W, \lhd)$ is congruence-free. To see this immediately, observe that the Rees congruence induced by the 0-minimal ideal $M(W, \lhd)$ is in fact the minimum proper congruence on $\operatorname{Res}(W, \lhd)$.

Another consequence of Theorem 1 is the following

PROPOSITION 1. *Let* $(V, \lhd')$ *and* $(W, \lhd)$ *be wosets with* 0 *and* 1. *If* θ: $V \to W$ *is a matrix isomorphism, then* F: $\operatorname{Res}(V, \lhd') \to \operatorname{Res}(W, \lhd)$ *defined by*

$$(11) \quad F(\psi) = \theta^{-1} \cdot \psi \cdot \theta \quad (\psi \in \operatorname{Res}(V, \lhd'))$$

is a (*semigroup*) *isomorphism. Conversely, any isomorphism* F: $\operatorname{Res}(V, \lhd') \to \operatorname{Res}(W, \lhd)$ *is induced by a unique matrix isomorphism* θ: $V \to W$ *such that* (11) *holds.*

Proof. The first part of the proposition has been proved in Section 3. The proof of the second assertion follows the usual lines (cf. [6, p. 28*f*; 10, Theorem 2; 11, Theorem 2]). Suppose that F: $\operatorname{Res}(V, \lhd') \to \operatorname{Res}(W, \lhd)$ is an isomorphism. Since $M(V, \lhd')$ and $M(W, \lhd)$ are densely embedded 0-minimal ideals, the restriction of F to $M(V, \lhd')$ is an isomorphism of $M(V, \lhd')$ onto $M(W, \lhd)$ which uniquely determines F. Thus there exist (unique) bijections θ and χ such that $F(\xi_u^v) = \xi_{u\theta}^{v\chi}$ ($u \neq 0$, $v \neq 1$ in V) and $0\theta = 0$, $1\chi = 1$. Since F and F^{-1} map nilpotent elements onto nilpotent elements, we get that $u \lhd' v \Leftrightarrow u\theta \lhd v\chi$ for all u, v. Hence θ is the required matrix isomorphism.

In view of Proposition 1 every automorphism F of the semigroup $\mathrm{Res}(W, \lhd)$ is inner—in the sense that F is induced by a matrix automorphism θ via (11). Therefore the automorphism group of $\mathrm{Res}(W, \lhd)$ is isomorphic to the group of all matrix automorphisms of $(W, \lhd)$. Recall that for a poset W this group coincides with the automorphism group of W. Note that for bounded posets W Proposition 1 follows from the results of Schein [11].

6. Maximal Submonoids of $\mathrm{Res}(W, \lhd)$

In this section we describe the maximal submonoids S of $\mathrm{Res}(W, \lhd)$ where $(W, \lhd)$ is any woset with 0 and 1. By definition, such a semigroup S is the maximum subsemigroup of $\mathrm{Res}(W, \lhd)$ having a given idempotent $\alpha^2 = \alpha \in \mathrm{Res}(W, \lhd)$ as an identity element, that is,

$$S = \alpha \cdot \mathrm{Res}(W, \lhd) \cdot \alpha = \{\varphi \in \mathrm{Res}(W, \lhd) | \alpha\varphi = \varphi\alpha = \varphi\}.$$

In other words, S is the semigroup of all maps $\varphi \in \mathrm{Res}(W, \lhd)$ such that $\mathrm{im}\, \varphi \subseteq \mathrm{im}\, \alpha$ and $\mathrm{im}\, \varphi^+ \subseteq \mathrm{im}\, \alpha^+$ (im denotes the image of a mapping). The intersection of S and $M(W, \lhd)$ is given by

$$\alpha \cdot M(W, \lhd) \cdot \alpha = \{\xi_b^a \mid a\alpha^+ = a \neq 1,\ b\alpha = b \neq 0\} \cup \{\xi_0\}.$$

By virtue of [8, Theorem 1] S is isomorphic to $\Omega(\alpha \cdot M(W, \lhd) \cdot \alpha)$. Hence the maximal submonoid S is isomorphic to the translational hull of a matrix 0-band $[\gamma]^0$ where γ is the restriction of $\not\rhd$ to $(\mathrm{im}\, \alpha^+ - \{1\}) \times (\mathrm{im}\, \alpha - \{0\})$. This matrix 0-band is reductive: γ is dense; and if for $u, v \in W$, $u\alpha \lhd x\alpha^+ \Leftrightarrow v\alpha \lhd x\alpha^+$ for all x, then $u\alpha \lhd x \Leftrightarrow v\alpha \lhd x$ for all x since (α, α^+) is an idempotent adjoint pair, whence $u\alpha = v\alpha$; similarly, $x\alpha \lhd u\alpha^+ \Leftrightarrow x\alpha \lhd v\alpha^+$ for all x implies $u\alpha^+ = v\alpha^+$. The matrix 0-band $[\gamma]^0 \cong \alpha \cdot M(W, \lhd) \cdot \alpha$ is not necessarily square. More precisely, we have

Theorem 2. *For a woset $(W, \lhd)$ with 0 and 1 every maximal submonoid of $\mathrm{Res}(W, \lhd)$ is isomorphic to the translational hull of some reductive matrix 0-band. Conversely, the translational hull of every reductive matrix 0-band is isomorphic to some maximal submonoid of a symmetric woset with ∞.*

Proof. It remains to verify the second assertion. Let $[\gamma]^0$ be any reductive matrix 0-band, where $\gamma \subseteq A \times B$ is a dense relation. We may assume that A and B are disjoint sets. Let $W = A \cup B \cup \{\infty\}$ such that $\infty \notin A \cup B$.

Define a symmetric relation $\perp$ on W as follows: (i) $\infty \perp x$ and $x \perp \infty$ for all $x \in W$, (ii) $x \perp y$ for all $x, y \in A$, (iii) $x \perp y$ for all $x, y \in B$, and (iv) for $a \in A$, $b \in B$, let $a \perp b$ and $b \perp a$ whenever $a\gamma b$ does not hold. It is readily checked that $(W, \perp)$ is a woset with ∞. Now define mappings $\alpha, \alpha^+ \colon W \to W$ by

$$x\alpha = \begin{cases} x & \text{if } x \in B \\ \infty & \text{otherwise} \end{cases} \quad \text{and} \quad y\alpha^+ = \begin{cases} y & \text{if } y \in A \\ \infty & \text{otherwise} \end{cases} \quad (x, y \in W).$$

We assert that for any $x, y \in W$, $x\alpha \perp y \Leftrightarrow x \perp y\alpha^+$. If $x \in W - B$, then $x\alpha \perp y$ and $x \perp y\alpha^+$ for all choices of y in W. Analogously, if $y \in W - A$, then $x\alpha \perp y$ and $x \perp y\alpha^+$ for all $x \in W$. Finally, if $x \in B$ and $y \in A$, then $x\alpha = x$ and $y\alpha^+ = y$. Therefore α and α^+ are adjoint. Further, α is idempotent and $A = \operatorname{im} \alpha^+ - \{\infty\}$, $B = \operatorname{im} \alpha - \{\infty\}$. Since γ is the restriction of the complement of $\perp$ to $A \times B$, we see that $\alpha \cdot M(W, \perp) \cdot \alpha$ is isomorphic to the given matrix 0-band $[\gamma]^0$, concluding the proof.

Particular interest attaches to the case where $(W, \leqq)$ is a bounded poset. Since by virtue of (7), (8) every (idempotent) residuated mapping α on W (i.e., $\alpha \in \operatorname{Res}(W, \leqq)$) restricts to an isomorphism of $\operatorname{im} \alpha^+$ onto $\operatorname{im} \alpha$, one can derive from [8, Theorem 1] that any maximal submonoid $\alpha \cdot \operatorname{Res}(W) \cdot \alpha$ is isomorphic to $\operatorname{Res}(\operatorname{im} \alpha)$. This is also true for arbitrary posets and, moreover, admits a straightforward proof.

PROPOSITION 2. *Let α be any idempotent residuated mapping on a poset W. Then the maximal submonoid $\alpha \cdot \operatorname{Res}(W) \cdot \alpha$ is isomorphic to* $\operatorname{Res}(\operatorname{im} \alpha)$. *If, in addition, W is bounded and α is bounded, then the maximal submonoid $\alpha \cdot \operatorname{Res}_{01}(W) \cdot \alpha$ of* $\operatorname{Res}_{01}(W)$ *is isomorphic to* $\operatorname{Res}_{01}(\operatorname{im} \alpha)$.

Proof. We shall freely use the fact that α satisfies (7), (8).

(i) Given $\varphi \in \alpha \cdot \operatorname{Res}(W) \cdot \alpha$, the restriction $\psi = \varphi|_{\operatorname{im} \alpha}$ of φ to $\operatorname{im} \alpha$ can be regarded as a self map of $\operatorname{im} \alpha$. For any $x, y \in W$ we have

$$\begin{aligned} (x\alpha)\,\psi = x\alpha\varphi \leqq y\alpha &\Leftrightarrow x\alpha \leqq y\alpha\varphi^+ \\ &\Leftrightarrow x\alpha = x\alpha\alpha^+\alpha \leqq (y\alpha)\,\varphi^+\alpha^+\alpha, \end{aligned}$$

whence $\psi \in \operatorname{Res}(\operatorname{im} \alpha)$ with $\psi^+ = \varphi^+\alpha^+\alpha|_{\operatorname{im} \alpha}$.

(ii) Given $\psi \in \operatorname{Res}(\operatorname{im} \alpha)$, the mapping $\varphi = \alpha\psi \colon W \to W$ has α as a two-sided identity. For $x, y \in W$ we get

$$\begin{aligned} x\varphi = x\alpha\psi \leqq y \Leftrightarrow (x\alpha)\,\psi = x\alpha\psi\alpha &= x\alpha\psi\alpha\alpha^+\alpha \leqq y\alpha^+\alpha \\ &\Leftrightarrow x\alpha \leqq (y\alpha^+\alpha)\,\psi^+ \\ &\Leftrightarrow x \leqq y\alpha^+\alpha\psi^+\alpha^+, \end{aligned}$$

whence $\varphi \in \alpha \cdot \operatorname{Res}(W) \cdot \alpha$ with $\varphi^+ = \alpha^+\alpha\psi^+\alpha^+$.

The correspondences $\varphi \mapsto \psi = \varphi|_{\operatorname{im}\alpha}$ of (i) and $\psi \mapsto \varphi = \alpha\psi$ of (ii) then establish an isomorphism $\varphi \leftrightarrow \psi$ between $\alpha \cdot \operatorname{Res}(W) \cdot \alpha$ and $\operatorname{Res}(\operatorname{im}\alpha)$.

The preceding arguments apply to the second statement of Proposition 2 as well (let all given maps α, φ, ψ be bounded).

Note that Proposition 2 and its proof generalize [1, Theorem 20.9, Part (1) $\Rightarrow$ (5), Corollary 2].

7. Applications to Semigroups of Binary Relations

A binary relation σ on a set X can be thought of as a mapping $\boldsymbol{\sigma}: \mathbf{2}^X \to \mathbf{2}^X$ on the lattice $\mathbf{2}^X$ of all subsets of X:

$$A\boldsymbol{\sigma} = A\sigma = \{x \in X \mid a\sigma x \text{ for some } a \in A\} \qquad (A \subseteq X).$$

Trivially, the image of $\boldsymbol{\sigma}$ coincides with the Zareckiĭ lattice $V(\sigma)$ of σ. Now, the mapping $\boldsymbol{\sigma}$ is residuated with residual $\boldsymbol{\sigma}^+$ given by

$$A\boldsymbol{\sigma}^+ = X - (X - A)\,\sigma^{-1} \qquad (A \subseteq X),$$

cf. [1, Exercise 4.15]. This is proved implicitly, for instance, in Zareckiĭ's paper [14]. Conversely, every residuated mapping on $\mathbf{2}^X$ is induced by a (unique) binary relation σ on X, cf. [1, Exercise 5.8]. Thus, $\sigma \to \boldsymbol{\sigma}$ sets up an isomorphism of the semigroup $\mathscr{B}(X)$ of all binary relations onto $\operatorname{Res}(\mathbf{2}^X)$. The 0-minimal ideal $\mathscr{K}(X)$ of $\mathscr{B}(X)$ consists of all rectangular binary relations, i.e., relations of the form $A \times B$ for some $A, B \subseteq X$. The residuated mapping ξ_b^a on $W = \mathbf{2}^X$ (see (9)) that corresponds to $A \times B \neq \varnothing$ is given by $a = X - A$ and $b = B$. Under the isomorphism $\mathscr{B}(X) \cong \operatorname{Res}(\mathbf{2}^X)$ any maximal submonoid of $\mathscr{B}(X)$ is isomorphic to $\alpha \cdot \operatorname{Res}(\mathbf{2}^X) \cdot \alpha \cong \operatorname{Res}(\operatorname{im}\alpha)$ for a suitable idempotent $\alpha \in \operatorname{Res}(\mathbf{2}^X)$. It is well known that a complete lattice L is isomorphic to the Zareckiĭ lattice $V(\sigma) = \operatorname{im}\boldsymbol{\sigma}$ of an idempotent binary relation σ on some set if and only if L is completely distributive (see Raney [9], cf. [14]). Hence the class of maximal submonoids of the semigroups of all binary relations coincides with the class of the semigroups of all residuated mappings on completely distributive complete lattices. The results of [16] are thus special instances of our results; for example, [16, Theorem 4.2] is a consequence of our Theorems 1 and 2.

One reason why we consider semigroups of bounded left adjoint maps in this paper is that among the we find all semigroups of dense relations. Indeed, a binary relation σ is dense if and only if the associated map $\boldsymbol{\sigma}$ is bounded. Therefore $\operatorname{Res}_{01}(\mathbf{2}^X)$ is isomorphic to the semigroup $\mathscr{D}(X)$ of all dense relations on X. The 0-minimal ideal of $\mathscr{D}(X)$ consists of $X \times X$ and the relations $A \times X \cup X \times B$ where A and B are subsets of X different from $\varnothing$, X. The corresponding map ζ_b^a on $W = \mathbf{2}^X$ (see (10)) is then given by

$a = X - A$ and $b = B$. Also, every completely distributive complete lattice L can be realized as im α for some bounded idempotent residuated map α on some $\mathbf{2}^X$. Hence all the results for $\mathscr{B}(X)$ mentioned here have analogues for $\mathscr{D}(X)$. For convenience, we record a concluding

COROLLARY 2. *The semigroup $\mathscr{D}(X)$ is isomorphic to $\Omega([\gamma]^0)$ where γ is the converse inclusion relation $\supseteq$ on $\mathbf{2}^X - \{\varnothing, X\}$. A semigroup S is isomorphic to a maximal submonoid of the semigroup of all dense relations on a set if and only if S contains a densely embedded ideal isomorphic to a matrix 0-band $[\gamma]^0$ where γ is the restriction of the partial order of a completely distributive complete lattice L to $L - \{0, 1\}$.*

An abstract characterization of $\mathscr{D}(X)$ in terms of the subsemigroup $\{\{u\} \times X \cup X \times \{u\} | u \in X\}$ was given in [15]. This result could also be used to prove our characterization of $\mathscr{D}(X)$.

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